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Wavelet Approximation of Periodic Functions

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We investigate expansions of periodic functions with respect to wavelet bases. Direct and inverse theorems for wavelet approximation in C and L_p norms are proved. For the functions possessing local regularity we study the rate of pointwise convergence of wavelet Fourier series. We also define and investigate the "discreet wavelet Fourier transform" (DWFT) for periodic wavelets generated by a compactly supported scaling function. The DWFT has one important advantage for numerical problems compared with the corresponding wavelet Fourier coefficients: while fast computational algorithms for wavelet Fourier coefficients are recursive, DWFTs can be computed by explicit formulas without any recursion and the computation is fast enough. © 2000 Academic Press

1. INTRODUCTION

In the past 10 years growing interest to wavelets can be explained mainly by their applications. However, it has turned out that some approximating properties of wavelet bases play an important role in analysis. For instance, there exist wavelets that constitute an unconditional basis in $L_p(\mathbb{R})$, $1<p<\infty$ (see, e.g., [3, Chap. 9]). The Meyer wavelets constitute an optimal basis in the space of continuous periodic functions [13, 14]. Certain functional classes can be described in terms of wavelet Fourier coefficients (see, e.g., [6, 12]). Connections between multiresolution approximation and structure properties of functions on $\mathbb{R}^d$ were studied in many publications (see, e.g., [5–9]). In the present paper we study wavelet bases as a tool for approximation of periodic functions.

We now introduce the necessary notation and definitions. Let $\mathbb{T}$ denote the unit circle. If $f\in L(\mathbb{T})$, then $\hat{f}(k)=\int_{\mathbb{T}} f(t)\, e^{-2\pi ikt}\, dt$ is kth Fourier coefficient of f (with respect to the trigonometric system). If $g\in L(\mathbb{R})\cup L_2(\mathbb{R})$, then $\hat{g}$ denotes the Fourier transform of g.

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Let φ be a function in $L_2(\mathbb{R})$ such that $\hat{\varphi}$ is bounded, $\lim_{t\to 0}\hat{\varphi}(t)=\hat{\varphi}(0)\neq 0$, for almost all $x\in\mathbb{R}$

$$\sum_{\ell\in\mathbb{Z}} |\hat{\varphi}(x+\ell)|^2=1, \tag{1}$$

and

$$\hat{\varphi}(x)=m_0(x/2)\,\hat{\varphi}(x/2), \tag{2}$$

where $m_0\in L_2(\mathbb{T})$. It is known (see, e.g., [3, Chap. 5]) that such a φ is a scaling function of a multiresolution analysis in $L_2(\mathbb{R})$ and there exists a function $\psi\in L_2(\mathbb{R})$ (wavelet function) such that its normalized integer shifts and scales $2^{j/2}\psi(2^j\cdot+n)$, $j, n\in\mathbb{Z}$, constitute an orthonormal basis in $L_2(\mathbb{R})$. The function ψ is defined by

$$\hat{\psi}(x)=m_0((x+1)/2)\,\hat{\varphi}(x/2)\,e^{\pi i x}. \tag{3}$$

If both the functions φ, ψ have sufficient decay, say

$$\max(|\varphi(x)|,|\psi(x)|)\leqslant\frac{\mathscr{C}}{1+|x|^{1+\varepsilon}},\qquad \varepsilon>0, \tag{4}$$

then the functions

$$\begin{aligned}\Phi_{jn}(x)&=2^{j/2}\sum_{\ell\in\mathbb{Z}}\varphi(2^jx+2^j\ell+n),\\ \Psi_{jn}(x)&=2^{j/2}\sum_{\ell\in\mathbb{Z}}\psi(2^jx+2^j\ell+n)\end{aligned} \tag{5}$$

are in $L_2(\mathbb{T})$ and the systems $\{\Phi_{jn}\}_{n=0}^{2^j-1}$, $\{\Psi_{jn}\}_{n=0}^{2^j-1}$ are orthonormal for each $j=0,1,2,\ldots$. Moreover, the spaces

$$\begin{aligned}V_j&=\operatorname{span}\{\Phi_{jn},\ n=0,\ldots,2^j-1\},\\ W_j&=\operatorname{span}\{\Psi_{jn},\ n=0,\ldots,2^j-1\}\end{aligned} \tag{6}$$

satisfy the properties

$$V_0=\{\text{const}\},\qquad V_j\subset V_{j+1},\qquad V_{j+1}=V_j\oplus W_j,\qquad \overline{\bigcup_{j=0}^{\infty}V_j}=L_2(\mathbb{T})$$

for all $j=0,1,\ldots$. This implies that $L_2(\mathbb{T})=V_0\oplus W_0\oplus W_1\oplus\cdots$. We shall call the collection $\{V_j\}_{j=0}^{\infty}$ a periodic multiresolution analysis (briefly PMRA) generated by φ. (Later, in Section 4, we shall introduce a wider class of

PMRA.) Thus, the functions Φ_{00}, Ψ_{jn}, $j=0, 1, ..., n=1, ..., 2^j-1$ (periodic wavelets) constitute an orthonormal basis in $L_2(\mathbb{T})$. Since under the assumption (4) these functions are also bounded, we can consider the wavelet Fourier series

$$\langle f, \Phi_{00}\rangle \, \Phi_{00} + \sum_{j=0}^{\infty} \sum_{n=0}^{2^j-1} \langle f, \Psi_{jn}\rangle \, \Psi_{jn} \tag{7}$$

for each $f \in L(\mathbb{T})$. To transform the double sum in (7) to a single one, we redenote wavelets: $w_0 = \Phi_{00}$, $w_{2^j+L} = \Psi_{jL}$, $0 \leqslant L < 2^j - 1$. Now (7) can be written as

$$\sum_{k=0}^{\infty} \langle f, w_k\rangle \, w_k. \tag{8}$$

Let $s_N(f)$ denote the Nth partial sum of (8). Since $s_{2^j-1}(f)$ is an orthogonal projection of f onto V_j, and $\{\Phi_{jn}\}_{n=0}^{2^j-1}$ is an orthonormal basis in V_j,

$$s_{2^j-1}(f) = \sum_{n=0}^{2^j} \langle f, \Phi_{jn}\rangle \, \Phi_{jn}, \tag{9}$$

and also each $s_N(f)$, $N = 2^j + L$, $0 \leqslant L < 2^j - 1$ can be represented in the form

$$s_N(f) = \sum_{n=0}^{2^j-1} \langle f, \Phi_{jn}\rangle \, \Phi_{jn} + \sum_{n=0}^{L} \langle f, \Psi_{jn}\rangle \, \Psi_{jn}. \tag{10}$$

Set $f = w_0 \equiv 1$ in (8). Since $\langle f, w_k\rangle = \delta_{0k}$, we have $s_N(f) = 1$ for all N, $j = 0, 1, ...$ Hence

$$\int_0^1 \sum_{k=0}^{N} w_k(x) \, \overline{w_k(t)} \, dt \equiv 1, \qquad \int_0^1 \sum_{k=0}^{2^j-1} \Phi_{jk}(x) \, \overline{\Phi_{jk}(t)} \, dt \equiv 1. \tag{11}$$

For a function $f \in L_p(\mathbb{T})$ ($f \in C(\mathbb{T})$ for $p = \infty$), we introduce the following characteristics: the rth modulus of smoothness $\omega_r(f, h)_p = \sup_{|t| \leqslant h} \|\Delta_t^r f\|_p$ and the error of best wavelet approximation of order N, $E_N(f)_p = \inf \|f - T\|_p$, where infinimum is taken over all "wavelet polynomials" $T = \sum_{k=0}^{N} \alpha_k w_k$.

Throughout the paper, $\mathscr{C}$ denotes a constant depending at most on a fixed PMRA and $\mathscr{C}(a, ..., b)$ denotes a constant depending at most on $a, ..., b$ and the PMRA.

2. DIRECT AND INVERSE APPROXIMATION THEOREMS

THEOREM 2.1. *Let* φ *satisfy* (4), $\psi \in C^{(m)}(\mathbb{R})$ *with* $\psi^{(\ell)}$ *bounded for* $\ell \leqslant m$, $|\psi(x)| \leqslant \mathscr{C}/(1+|x|^n)$, $n > m+1$, $p \in [1, \infty]$. *Then*

$$E_N(f)_p \leqslant \|f - s_N(f)\|_p \leqslant \mathscr{C}(p, n, m)\, \omega_r \left(f, \frac{1}{N}\right)_p, \qquad N = 1, 2, \ldots \quad (12)$$

for all $f \in L_p(\mathbb{T})$ ($f \in C(\mathbb{T})$ *for* $p = \infty$) *and for all positive integers* r, $r \leqslant m+1$, $r < n-1$.

This theorem is a wavelet analog of the classical Jackson's theorem for the trigonometric approximation. However, in contrast to the trigonometric case, not only the errors of best approximation but also the deviations of functions from partial sums of their wavelet Fourier series are majorized by moduli of smoothness.

Theorem 2.1 is known [13, 14] for Meyer wavelets. In this case $\|f - s_N(f)\|$ is majorized by the error of trigonometric best approximation moreover, it is majorized by the modulus of smoothness of arbitrary order. The proof of this result was based on very special properties of Meyer wavelets.

Before giving the proof of Theorem 2.1 we shall present three simple auxiliary statements.

LEMMA 2.2. *Let* g, h *be functions defined on* $\mathbb{R}$, $\max(|g(x)|, |h(x)|) \leqslant \mathscr{C}/(1+|x|^{1+\varepsilon})$, $\varepsilon > 0$, *and let* $f \in L(\mathbb{T})$, $j = 0, 1, \ldots, L = 0, \ldots, 2^j - 1$. *Then*

$$\int_0^1 f(t) \sum_{k=0}^{L} \sum_{\ell' \in \mathbb{Z}} g(2^j x + 2^j \ell' + k) \sum_{\ell \in \mathbb{Z}} \overline{h(2^j t + 2^j \ell + k)}\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \sum_{v \in \mathbb{Z}(j, L)} g(2^j x + v)\, \overline{h(2^j t + v)}\, dt, \qquad (13)$$

where $\mathbb{Z}(j, L) = \{v \in \mathbb{Z} : v = 2^j \ell + k, \ell \in \mathbb{Z}, k = 0, \ldots, L\}$.

The proof of this lemma is trivial.

LEMMA 2.3. *Let* μ *be a bounded, decreasing, and integrable function on* $[0, \infty)$. *Then for all* $x, y \in \mathbb{R}$

$$\sum_{k \in \mathbb{Z}} \mu(|x+k|)\, \mu(|y+k|) \leqslant C \mu\left(\frac{|x-y|}{4}\right),$$

where C *is a constant depending only on* μ.

The proof of this lemma is simple, a little bit more general statements are presented in [8, 16].

COROLLARY 2.4. *If g, h satisfy the hypothesis of Lemma 2.2, then*

$$2^j \int_0^1 \left| \sum_{k=0}^{L} \sum_{\ell' \in \mathbb{Z}} g(2^j x + 2^j \ell' + k) \sum_{\ell \in \mathbb{Z}} \overline{h(2^j t + 2^j \ell + k)} \right| dt \leqslant C, \tag{14}$$

where C is a constant depending only on the functions g, h, and ε.

To prove (14), we should apply Lemma 2.2 for $f \equiv 1$ and Lemma 2.3 for $\mu(x) = 1/(1+|x|^{1+\varepsilon})$.

Proof of Theorem 2.1. First we assume that f is a trigonometric polynomial and prove the following inequality:

$$\|f - s_N(f)\|_p \leqslant \mathscr{C}(p, r, n) \frac{\|f^{(r)}\|_p}{N^r}. \tag{15}$$

Let $2^j \leqslant N \leqslant 2^{j+1}$, since $\lim_{k\to\infty} \|f - s_{2^k-1}(f)\|_p = 0$ (this fact is well known; it also can be easily deduced from (9) and (14)),

$$f - s_N(f) = (s_{2^j-1}(f) - s_N(f)) + \sum_{i=j}^{\infty} (s_{2^{i+1}-1}(f) - s_{2^i-1}(f)). \tag{16}$$

Hence, to prove (15), it suffices to check that

$$\|s_{2^j+L}(f) - s_{2^j-1}(f)\|_p \leqslant \mathscr{C}(p, r, n)\, 2^{-jr} \|f^{(r)}\|_p \tag{17}$$

for all $j = 0, 1, ..., L = 0, ..., 2^j - 1$. Using Lemma 2.2, the Taylor formula

$$f(t) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x)}{k!} (t-x)^k + \frac{1}{(r-1)!} \int_x^t f^{(r)}(\tau)(t-\tau)^{r-1}\, d\tau,$$

and the equalities

$$\int_{-\infty}^{\infty} x^\ell \psi(x)\, dx = 0, \qquad \ell = 0, ..., m \tag{18}$$

(see, e.g., [3, Chap. 5]), we have

$$\begin{aligned} &s_{2^j+L}(f, x) - s_{2^j-1}(f, x) \\ &\quad = 2^j \int_0^1 f(t) \sum_{k=0}^{L} \sum_{\ell' \in \mathbb{Z}} \psi(2^j x + 2^j \ell' + k) \sum_{\ell \in \mathbb{Z}} \overline{\psi(2^j t + 2^j \ell + k)}\, dt \\ &\quad = 2^j \int_{-\infty}^{\infty} f(t) \sum_{\nu \in \mathbb{Z}(j, L)} \psi(2^j x + \nu) \overline{\psi(2^j t + \nu)}\, dt \\ &\quad = \frac{2^j}{(r-1)!} \int_{-\infty}^{\infty} \int_x^t f^{(r)}(\tau)(t-\tau)^{r-1}\, d\tau \sum_{\nu \in \mathbb{Z}(j, L)} \psi(2^j x + \nu) \overline{\psi(2^j t + \nu)}\, dt. \end{aligned}$$

It is possible to change the order of summation and integration, by Lebesgue's theorem. Set $\mu(u)=1/(1+|u|^n)$. By Lemma 2.3,

$$\left|\sum_{\nu\in\mathbb{Z}(j,L)} \psi(u+\nu)\,\overline{\psi(v+\nu)}\,dt\right| \leqslant \mathscr{C}(n)\,\mu(u-v)$$

for all $u, v\in\mathbb{R}$. Hence

$$\begin{aligned}
&|s_{2^j+L}(f,x)-s_{2^j-1}(f,x)|\\
&\quad\leqslant \mathscr{C}(n)\,2^j\int_{-\infty}^{\infty}\left|\int_x^t f^{(r)}(\tau)(t-\tau)^{r-1}\,d\tau\right|\mu(2^j(t-x))\,dt\\
&\quad= \mathscr{C}(n)\,2^j\int_{-\infty}^{\infty}\left|\int_x^{x+t} |f^{(r)}(\tau)|\,|t+x-\tau|^{r-1}\,d\tau\right|\mu(2^j t)\,dt. \qquad (19)
\end{aligned}$$

Applying Jensen's inequality for $p<\infty$, we obtain from (19)

$$\begin{aligned}
&\|s_{2^j+L}(f)-s_{2^j}(f)\|_p\\
&\quad\leqslant 2^j\int_{-\infty}^{\infty} dt\,\mu(2^j t)\left(\int_0^1\left|\int_x^{x+t}|f^{(r)}(\tau)|\,|x+t-\tau|^{r-1}\,d\tau\right|^p dx\right)^{1/p}\\
&\quad\leqslant \mathscr{C}(p,r,n)\,2^j\int_{-\infty}^{\infty} dt\,\mu(2^j t)\\
&\qquad\times\left(\int_0^1 |t|^{r(p-1)}\left|\int_x^{x+t}|f^{(r)}(\tau)|^p\,|x+t-\tau|^{r-1}\,d\tau\right| dx\right)^{1/p}. \qquad (20)
\end{aligned}$$

If $|t|\geqslant 1$, then taking into account the periodicity of $f^{(r)}$, we have

$$\left|\int_x^{x+t}|f^{(r)}(\tau)|^p\,|x+t-\tau|^{r-1}\,d\tau\right|\leqslant |t|^{r-1}\left|\int_x^{x+t}|f^{(r)}|^p\right|\leqslant 2\,|t|^r\,\|f^{(r)}\|_p^p.$$

This implies that

$$\int_0^1 dx\left|\int_x^{x+t}|f^{(r)}(\tau)|^p\,|x+t-\tau|^{r-1}\,d\tau\right|\leqslant \mathscr{C}(r)\,|t|^r\,\|f^{(r)}\|_p^p. \qquad (21)$$

If $0\leqslant t<1$, then

$$\begin{aligned}
&\int_0^1 dx\int_x^{x+t}|f^{(r)}(\tau)|^p\,|x+t-\tau|^{r-1}\,d\tau\\
&\qquad\leqslant \int_0^{t+1}|f^{(r)}(\tau)|^p\,d\tau\int_{\tau-t}^{\tau}(t+x-\tau)^{r-1}\,dx\leqslant \frac{2}{r}\,\|f^{(r)}\|_p^p\,t^r.
\end{aligned}$$

Thus, (21) holds also in this case. Similarly, if $-1<t<0$, then

$$\left|\int_x^{x+t} |f^{(r)}(\tau)|^p \, |x+t-\tau|^{r-1} \, d\tau\right| \leqslant \int_t^1 |f^{(r)}(\tau)|^p \, d\tau \int_\tau^{\tau-t} (\tau-x-t)^{r-1} \, dx \leqslant \frac{2}{r} \, \|f^{(r)}\|_p^p \, |t|^r.$$

Again this implies (21).

Substituting (21) into (20) yields

$$\|s_{2^j+L}(f)-s_{2^j-1}(f)\|_p \leqslant \mathscr{C}(p, r, n) \, \|f^{(r)}\|_p \, 2^j \int_{-\infty}^{\infty} |t|^r \, \mu(2^j t) \, dt$$

for $p<\infty$. For $p=\infty$, the inequality

$$\|s_{2^j+L}(f)-s_{2^j-1}(f)\|_\infty \leqslant \mathscr{C}(r, n) \, \|f^{(r)}\|_\infty \, 2^j \int_{-\infty}^{\infty} |t|^r \, \mu(2^j t) \, dt.$$

evidently follows from (19). These relations imply (15) immediately, whenever we change variable in the integrals and take into account that the functions μ, $|t|^r \mu(t)$ are summable on $\mathbb{R}$. Hence, (15) holds for all trigonometric polynomials. Thus (15) implies (12) for each trigonometric polynomial f, due to the following theorem of Zhuk [19].

THEOREM 2.5. *Let Φ be a non-negative semi-additive functional on $L_p(\mathbb{T})$ $(C(\mathbb{T}))$, $Y_{r,n}=\sup(\Phi(g)/\|g^{(r)}\|_p)$, where the supremum is taken over all trigonometric polynomials g of order n, $Y_r=\sup_k Y_{r,k}$. Then for all positive integers N, r and all trigonometric polynomials f*

$$\Phi(f) \leqslant A_r(1+Y_0+N^r Y_{rN}) \, \omega_r\left(f, \frac{1}{N}\right)_p,$$

where A_r is a constant depending only on r.

To prove Theorem 2.1 for an arbitrary function $f \in L_p(\mathbb{T})$ ($f \in C(\mathbb{T})$ for $p=\infty$) it remains to approximate f in the norm by a trigonometric polynomial f_1, to apply (2.1) to f_1, and to take into account that $\|s_N(f)\|_p \leqslant \mathscr{C}(p) \, \|f\|_p$, $\omega_r(f, \frac{1}{N}) \leqslant \mathscr{C}(r) \, \|f\|_p$.

Remark. In the hypotheses of Theorem 2.1 the smoothness of ψ can be replaced by (18).

THEOREM 2.6. *Let $\varphi \in C^{(m)}(\mathbb{R})$ satisfy (4) and $|\varphi^{(m)}(x)| \leqslant \mathscr{C}/(1+|x|^{1+\varepsilon})$, $\varepsilon>0$, $p \in [1, \infty]$. Then*

$$\|f^{(m)}\|_p \leqslant \mathscr{C}(p, m) \, 2^{mj} \, \|f\|_p, \tag{22}$$

for all $f \in V_j$, $j=0, 1, \ldots$.

This theorem is a wavelet analog of Bernstein's inequality for trigonometric polynomials.

Proof. Let $f \in V_j$. Since $f = s_{2^j-1}(f)$, by (9), we have

$$f(x) = 2^j \int_0^1 f(t) \sum_{k=0}^{2^j-1} \sum_{\ell' \in \mathbb{Z}} \varphi(2^j t + 2^j \ell' + k) \sum_{\ell \in \mathbb{Z}} \overline{\varphi(2^j x + 2^j \ell + k)}\, dt. \tag{23}$$

Hence

$$f^{(m)}(x) = 2^{j(m+1)} \int_0^1 f(t)\, K_j(x, t)\, dt, \tag{24}$$

where

$$K_j(x, t) = \sum_{k=0}^{2^j-1} \sum_{\ell' \in \mathbb{Z}} \varphi^{(m)}(2^j x + 2^j \ell' + k) \sum_{\ell \in \mathbb{Z}} \overline{\varphi(2^j t + 2^j \ell + k)}.$$

For $p = \infty$ (22) follows from this immediately, due to Corollary 2.4 with $g = \varphi$, $h = \varphi^{(m)}$. Consider $p < \infty$. By Jensen's inequality, (24) implies

$$\| f^{(m)} \|_p^p \leqslant 2^{j(m+1)} \int_0^1 dx \left(\int_0^1 |K_j(x, t)|\, dt \right)^{p-1} \int_0^1 |f(t)|^p\, |K_j(x, t)|\, dt.$$

Finally, applying Corollary 2.4, we obtain (22). ∎

THEOREM 2.7. *Let φ satisfy the hypothesis of Theorem 2.6, $p \in [1, \infty]$. Then*

$$\omega_m(f, h)_p \leqslant \mathscr{C}(p, m)\, h^m \sum_{0 \leqslant \ell \leqslant h^{-1}} (\ell + 1)^{m-1} E_\ell(f)_p$$
$$\leqslant \mathscr{C}(p, m)\, h^m \sum_{0 \leqslant \ell \leqslant h^{-1}} (\ell + 1)^{m-1} \| f - s_\ell(f) \|_p$$

for all $h > 0$ and $f \in L_p(\mathbb{T})$ ($f \in C(\mathbb{T})$ for $p = \infty$). Moreover, if

$$\sum_{\ell=1}^{\infty} \ell^{m-1} \| f - s_\ell(f) \|_p < \infty,$$

then the function f has a derivative of order m almost everywhere (at each point in the case $p = \infty$), $f^{(m)} \in L_p(\mathbb{T})$ ($f^{(m)} \in C(\mathbb{T})$ for $p = \infty$), and

$$\| f^{(m)} - s_n^{(m)}(f) \|_p \leqslant \mathscr{C}(p, m) \sum_{\ell = [n/2]}^{\infty} (\ell + 1)^{m-1} \| f - s_\ell(f) \|_p.$$

To prove this theorem we should repeat the proof of the similar statement for trigonometric polynomials (see, e.g., [11]) using Theorem 2.6 instead of the classical Bernstein's inequality.

The hypotheses of both Theorem 2.1 and Theorem 2.7 hold for the wavelets generated by a smooth compactly supported scaling function. Such wavelets were constructed by Daubechies (see [3, Chap. 6]). For these wavelets Theorems 2.1 and 2.7 give the following statement.

COROLLARY 2.8. *Let* $\varphi, \psi \in C^{(m)}(\mathbb{R})$ *be compactly supported with* $\psi^{(\ell)}$ *bounded for* $\ell \leqslant m$, *and let* $r \leqslant m-1$ *be a non-negative integer,* $\alpha \in [0, 1)$, $\alpha^2 + r^2 > 0$, $1 \leqslant p \leqslant \infty$, $f \in L_p(\mathbb{T})$, $(f \in C(\mathbb{T})$ *for* $p = \infty)$. *Then the relations* $E_N(f)_p = O(1/N^{r+\alpha})$, $\|f - s_N(f)\|_p = O(1/N^{r+\alpha})$, $\omega_{r+1}(f, h)_p = O(h^{r+\alpha})$ *are equivalent.*

COROLLARY 2.9. *Let numbers* r, α *and functions* φ, ψ *satisfy the conditions of Corollary* 2.8 *and let* $f \in C(\mathbb{T})$. *The relation*

$$\omega_{r+1}(f, h)_\infty = O(h^{r+\alpha}) \tag{25}$$

holds if and only if

$$\langle f, w_N \rangle = O(2^{-j(r+\alpha+1/2)}) \tag{26}$$

for all $N = 2^j + n$, $n = 0, ..., 2^j - 1$.

Proof. Due to Theorem 2.1 and the evident relation $\|w_N\|_1 = O(2^{-j/2})$, $N = 2^j + n$, $n = 0, ..., 2^j - 1$, we have (25) $\Rightarrow$ (26). It follows from Corollary 2.8 and (16) that (25) is equivalent to $\|s_{2^j+\ell} - s_{2^j-1}\|_\infty = O(2^{-j(r+\alpha)})$, $\ell = 0, ..., 2^j - 1$. If (26) holds, then $\|s_{2^j+\ell} - s_{2^j-1}\|_\infty = O(2^{-j(r+\alpha+1/2)} \times \|\sum_{k=0}^{\ell} |w_k|\|_\infty)$. To prove (26) $\Rightarrow$ (25) it remains to note that $\sum_{k=0}^{\ell} |w_k| = O(2^{j/2})$ because of the compactness of the support of ψ. ∎

COROLLARY 2.10. *Let a number* r *and functions* φ, ψ *satisfy the conditions of Corollary* 2.8, $1 \leqslant p \leqslant \infty$, $0 < \alpha < r$, $0 < q < \infty$. *A function* $f \in L_p(\mathbb{T})$ $(f \in C(\mathbb{T})$ *for* $p = \infty)$ *belongs to the Besov space* $B_q^\alpha(L_p)$ *if and only if* $(\sum_{n=1}^{\infty} \frac{1}{n} (n^\alpha \|f - s_{n-1}(f)\|_p)^q)^{1/q} < \infty$.

This statement follows immediately from Theorems 2.1, 2.6, due to Theorem 9.1 from [4].

3. LOCAL CONVERGENCE

In this section we study wavelet approximation of functions possessing local regularity.

Let $x_0 \in \mathbb{R}$, $\alpha > 0$. We say that a function $f \in L(\mathbb{T})$ belongs to $\mathcal{L}_\alpha(x_0)$ if there exists a number s such that

$$\frac{1}{h}\int_{|x-x_0|\leqslant h} |f(x)-s|\,dx = O(h^\alpha), \qquad \text{as} \quad h\to 0. \tag{27}$$

THEOREM 3.1. *Let* $f\in\mathcal{L}_\alpha(x_0)$, $\alpha>0$. *If* $|\varphi(x)|\leqslant \mathscr{C}/(1+|x|^m)$, $m>1$, *then*

$$|s_{2^j-1}(f,x_0)-s| = O(2^{-j\min\{\alpha,\,m-1\}}), \qquad j\to\infty, \tag{28}$$

for $\alpha\neq m-1$;

$$|s_{2^j-1}(f,x_0)-s| = O(j2^{-j\alpha}), \qquad j\to\infty, \tag{29}$$

for $\alpha = m-1$. *If, moreover,* $|\psi(x)|\leqslant\mathscr{C}/(1+|x|^n)$, $n>1$, *then*

$$|s_N(f,x_0)-s| = O(N^{-\min\{\alpha,\,n-1\}}), \qquad N\to\infty, \tag{30}$$

for $\alpha\neq n-1$;

$$|s_N(f,x_0)-s| = O(N^{-\alpha}\log N), \qquad N\to\infty, \tag{31}$$

for $\alpha = n-1$.

Proof. It follows from (9), (11) that

$$s_{2^j-1}(f,x_0)-s = \int_0^1 (f(t)-s)\sum_{n=0}^{2^j-1}\Phi_{jn}(t)\,\overline{\Phi_{jn}(x_0)}\,dt.$$

Hence, by Lemmas 2.2, 2.3,

$$|s_{2^j-1}(f,x_0)-s|\leqslant 2^j\int_{-\infty}^{\infty}|f(t)-s|\,\mu(2^j(t-x_0))\,dt, \tag{32}$$

where $\mu(u)=\mathscr{C}/(1+|u|^m)$. Let j_0 be a positive integer such that

$$\frac{1}{h}\int_{|x-x_0|\leqslant h}|f(x)-s|\,dx\leqslant\mathscr{C}h^\alpha, \qquad 0<h\leqslant 2^{-j_0}. \tag{33}$$

Due to the monotonicity of μ, for all $j\geqslant j_0$

$$\begin{aligned}
&|s_{2^j-1}(f,x_0)-s|\\
&\quad\leqslant 2^{j+1}\Bigg(\mu(0)\int_{|t-x_0|<2^{-j}}|f(t)-s|\,dt+\sum_{k=-j}^{\infty}\mu(2^{j+k})\\
&\qquad\times\int_{2^k\leqslant|t-x_0|\leqslant 2^{k+1}}|f(t)-s|\,dt\Bigg)
\end{aligned}$$

$$\leqslant 2^{j+1}\mu(0)\int_{|t-x_0|<2^{-j}}|f(t)-s|\,dt$$

$$+\sum_{k=-j+1}^{-j_0} 2^{j+1}\mu(2^{j+k})\int_{|t-x_0|<2^k}|f(t)-s|\,dt$$

$$+\sum_{k=-j_0+1}^{\infty} 2^{j+1}\mu(2^{j+k})\int_{|t-x_0|<2^k}|f(t)-s|\,dt=\Sigma_0+\Sigma_1+\Sigma_2. \tag{34}$$

Using (3.1) we have

$$\Sigma_0=O(2^{-j\alpha})=O(2^{-j\min\{\alpha,\,m-1\}}),$$
$$\Sigma_1=O(1)\,2^j\sum_{k=-j}^{-j_0}2^{k(\alpha+1)}2^{-(j+k)m}=O\left(2^{j(1-m)}\sum_{k=-j+1}^{-j_0}2^{k(\alpha+1-m)}\right). \tag{35}$$

This gives

$$\Sigma_1=O(2^{-j\min\{\alpha,\,m-1\}}),\qquad \text{if}\quad \alpha\neq m-1, \tag{36}$$

$$\Sigma_1=O(j2^{-j\alpha}),\qquad \text{if}\quad \alpha=m-1. \tag{37}$$

To estimate Σ_2, we note that the function $I(h)=\frac{1}{h}\int_{|t-x_0|<h}|f(t)-s|\,dt$ is bounded on $(0,\infty)$. This implies that

$$\Sigma_2=O(1)\sum_{k=-j_0+1}^{\infty}2^{j+k}\mu(2^{j+k})=O(1)\sum_{k=-j_0+1}^{\infty}2^{(j+k)(1-m)}$$

$$=O(1)\,2^{j(1-m)}\sum_{k=-j_0+1}^{\infty}2^{k(1-m)}=O(2^{-j(m-1)})=O(2^{-j\min\{\alpha,\,m-1\}}). \tag{38}$$

Combining (35), (36), (37), (38) with (34), we obtain (28), (29). In particular, (28) or (29) implies that the sequence $\{s_{2^j-1}(f,x_0)\}_{j=1}^{\infty}$ converges to s. From this it follows that (16) holds at the point x_0. Hence to prove (30), (31) it suffices to establish the relations

$$s_{2^j+L}(f,x_0)-s_{2^j-1}(f,x_0)=O(2^{-j\min\{\alpha,\,m-1\}}),\qquad \text{if}\quad \alpha\neq m-1, \tag{39}$$

$$s_{2^j+L}(f,x_0)-s_{2^j-1}(f,x_0)=O(j2^{-j\alpha}),\qquad \text{if}\quad \alpha=m-1, \tag{40}$$

for all $j=0,1,...,L=0,...,2^j-1$.

Since, due to (11), the left-hand sides of these equalities can be represented as

$$\int_0^1 (f(t)-s) \sum_{k=0}^{L} \Psi_{jk}(t)\, \overline{\Psi_{jk}(x_0)}\, dt,$$

(39), (40) can be proved similarly to (28), (29). ▮

Next we shall show that estimates (28), (29), (30), (31) cannot be improved in general. Consider the special case $\alpha < n-1$, $\alpha < m-1$ which holds, for example, if the functions φ, ψ are compactly supported. In this case (28) and (30) look like

$$|s_N(f, x_0) - s| = O(N^{-\alpha}), \qquad N \to \infty. \tag{41}$$

The following example illustrates that (41) is sharp. Consider the Haar system that is a wavelet system with

$$\varphi(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \psi(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < 1/2, \\ -1, & \text{if } 1/2 \leqslant x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

Set $f(t) = t^{\alpha}$ on $[0, \delta]$, $f(x) = 0$ on $[-\delta, 0]$. It is clear that $f \in L(\mathbb{T})$ and satisfies (27) with $s = 0$, $x_0 = 0$. Consider the $(2^j - 1)$th partial sums:

$$s_{2^j-1}(f, x) = 2^j \int_0^1 ff(t) \sum_{k=0}^{2j-1} \sum_{\ell \in \mathbb{Z}} \varphi(2^j x + 2^j \ell + k) \sum_{\ell' \in \mathbb{Z}} \overline{\varphi(2^j t + 2^j \ell' + k)}\, dt.$$

If $x \in [0, 2^{-j})$, then $\sum_{\ell \in \mathbb{Z}} \varphi(2^j x + 2^j \ell) = 1$ and $\sum_{\ell \in \mathbb{Z}} \varphi(2^j x + 2^j \ell + k) = 0$ for $k = 1, ..., 2^j - 1$. This implies that for all j large enough

$$|s_{2^j-1}(f, 0)| = 2^j \int_0^{2^{-j}} f(t)\, dt = 2^j \int_0^{2^{-j}} t^{\alpha}\, dt = \frac{1}{\alpha} 2^{-j\alpha}.$$

Thus, the right-hand side of (41) can not be replaced by $o(N^{-\alpha})$.

Now we consider the case $n - 1 < \alpha$. In this case

$$|s_N(f, x_0) - s| = O(N^{(1-n)}), \qquad N \to \infty.$$

In particular, the following "localization principle" holds: if $f \equiv 0$ on a neighborhood of a point x_0, then (43) is valid. There exists an example [17] illustrating that (43) is sharp.

The following example illustrates that the estimates (29), (31) are sharp. Define φ by

$$\hat{\varphi}(u)=\begin{cases} 1, & \text{if} \quad |u|<1/3, \\ \left|\sin\dfrac{3\pi u}{2}\right|, & \text{if} \quad 1/3<|u|<2/3 \\ 0, & \text{if} \quad |u|>2/3. \end{cases}$$

This is a scaling function of Meyer non-smooth wavelets (see, e.g., [3, Chap. 5]), It is not difficult to compute φ explicitly:

$$\varphi(u)=\frac{\sin 2\pi u/3}{2\pi u}+\frac{6\cos 4\pi u/3+8u\sin 2\pi u/3}{\pi(9-16u^2)}.$$

It is clear that $\varphi(u)$ decays as u^{-2}. We can also compute ψ, due to the equality $\hat{\psi}(u)=e^{i\pi u}(\hat{\varphi}(u+1)+\hat{\varphi}(u-1))\,\hat{\varphi}(u/2)$ (see [3, Chap. 5]), and verify that $\psi(u)=O(u^{-2})$. Thus, φ generates a PMRA and the functions φ, ψ satisfy the hypothesis of Theorem 3.1 with $n=m=2$. We shall prove that for any $\gamma(u)=o(u\log u)$, $u\to+0$ there exists a 1-periodic function $f\in L(\mathbb{T})$ satisfying (27) with $x_0=0$, $s=0$, $\alpha=1$ such that $\limsup_{j\to\infty}|\gamma^{-1}(2^{-j})\times s_{2^j-1}(f,0)|=\infty$. Due to the Banach–Steinhaus theorem, it suffices to find a sequence of positive integers $\{M_j\}_j$, $M_j\to\infty$ and a sequence of functions $\{f_j\}_j$, $f_j\in L(\mathbb{T})$, such that

$$\sup_{h>0}\frac{1}{h^2}\int_{-h}^{h}|f_j|\leqslant C, \tag{44}$$

where C is an absolute constant, and $|\gamma^{-1}(2^{-j})\,s_{2^j-1}(f_j,0)|\geqslant M_j$ for some subsequence of positive integers j. Consider a sequence of positive integers $N_j=2^j-1$ with even j. Using the Poisson summation formula, we have

$$\sum_{\ell\in\mathbb{Z}}\varphi(2^j(x+\ell)+k)=\sum_{n\in\mathbb{Z}}\hat{\varphi}(2^{-j}n)\,e^{2\pi inx}\,e^{2\pi ink2^{-j}}.$$

Then taking into account that $\sum_{k=0}^{2^j-1}e^{2\pi ikm2^{-j}}=0$ for all $m\neq 2^j\ell$, $\ell\in\mathbb{Z}$, and $\operatorname{supp}\hat{\varphi}\cap\operatorname{supp}\hat{\varphi}(\cdot+k)=0$ for $k\in\mathbb{Z}$, $|k|\geqslant 2$, we obtain

$$\begin{aligned} s_{N_j}(f,x)=\int_0^1 f(t)\sum_{n\in\mathbb{Z}}\hat{\varphi}(2^{-j}n)\,e^{2\pi in(t-x)}(\hat{\varphi}(2^{-j}n)+\hat{\varphi}(2^{-j}n-1)\,e^{2\pi i2^jx} \\ +\hat{\varphi}(2^{-j}n+1)\,e^{-2\pi i2^jx})\,dt \end{aligned}$$

for all $f\in L(\mathbb{T})$. This implies that

$$s_{N_j}(f,0)=\sum_{n\in\mathbb{Z}}\lambda(2^{-j}n)\,\hat{f}(n),$$

where $\lambda(u)=\hat{\varphi}^2(u)+\hat{\varphi}(u)(\hat{\varphi}(u-1)+\hat{\varphi}(u+1))$. It is not difficult to verify that $\hat{\lambda}(u)=\alpha(u)+\beta(u)$, where

$$\alpha(u)=\frac{9(\sin 2\pi u/3+\sin 4\pi u/3)}{2\pi u(9-4u^2)},$$

$$\beta(u)=\frac{3(\cos 2\pi u/3+\cos 4\pi u/3)}{\pi(9-4u^2)}.$$

Set $\sigma_N(f,x)=\sum_{n\in\mathbb{Z}}\lambda(\frac{n}{N})\,\hat{f}(n)\,e^{2\pi inx}$. This is a linear summation method of Fourier series with summable $\hat{\lambda}$. It is well known that $\sigma_N(f)$ can be represented in the following form:

$$\sigma_N(f,x)=\int_{-\infty}^{\infty} f\left(x+\frac{t}{N}\right)\hat{\lambda}(t)\,dt.$$

Since $\sigma_{2^j}(f,0)=s_{N_j}(f,0)$, we obtain

$$s_{N_j}(f,0)=\int_{-\infty}^{\infty} f(2^{-j}t)\,\hat{\lambda}(t)\,dt. \tag{45}$$

Set $\omega=\{k_r=2^{r+j/2},\ldots,2^{r+j/2}+2^{2r-1},\ r=1,\ldots,\frac{j}{2}-3\}$, $e_k=[3k,3k+1/2]$, $k\in\mathbb{Z}$, $\Omega=\bigcup_{k\in\omega}e_k$. It is not difficult to see that $\Omega\subset[0,2^{j-1}]$ and $e_k\cap e_\ell=\varnothing$ for all $k,\ell\in\omega$, $k\neq\ell$, whenever $j\geqslant 4$. Introduce even 1-periodic functions f_j defined on $[0,1/2]$ by

$$f_j(x)=\begin{cases}-1, & \text{if}\quad 2^jx\in\Omega,\\ 0, & \text{if}\quad 2^jx\in[0,2^{j-1}]\backslash\Omega.\end{cases}$$

Since $f_j\equiv 0$ on $[0,2^{1-j/2}]$, (44), evidently, holds for all $h\leqslant 2^{1-j/2}$. If $2^{\ell-j/2}\leqslant h<2^{\ell+1-j/2}$, $1\leqslant\ell\leqslant\frac{j}{2}-4$, we have

$$\int_0^h|f_j|=\int_{\Omega\cap[0,h]}|f_j|\leqslant\sum_{r=1}^{\ell}\ \sum_{k_r=2^{r+j/2}}^{2^{2r-1}+2^{r+j/2}}2^{-j-1}$$
$$\leqslant 2^{-j-1}\sum_{r=1}^{\ell}2^{2r}=O(2^{2\ell-j})=O(h^2).$$

It is clear that

$$\int_{-\infty}^{\infty}f_j(2^{-j}t)\,\hat{\lambda}(t)\,dt=\int_{-\infty}^{\infty}f_j(2^{-j}t)\,\beta(t)\,dt+O(2^{-j})$$
$$=-\frac{3}{2\pi}\int_0^{\infty}f_j(2^{-j}t)\,P(t)\,\frac{dt}{t^2}+O(2^{-j}),$$

where $P(t)=\cos(2\pi t/3)+\cos(4\pi t/3)$. Further,

$$-\int_0^\infty f_j(2^{-j}t)\,P(t)\,\frac{dt}{t^2}=\int_\Omega P(t)\,\frac{dt}{t^2}+O(2^{-j}).$$

Since $P(t)$ is 3-periodic and non-negative on Ω, for all even $j>6$

$$\begin{aligned}\int_\Omega P(t)\,\frac{dt}{t^2}&=\sum_{r=1}^{j/2-3}\sum_{k_r=2^{r+j/2}}^{2^{r+j/2}+2^{2r-1}}\int_{e_{k_r}} P(t)\,\frac{dt}{t^2}\geqslant\sum_{r=1}^{j/2-3}\int_{2^{r+j/2}+1}^{2^{r+j/2}+2^{2r-1}+1}\frac{dt}{9t^2}\int_{e_0}P\\&\geqslant\frac{1}{9}\sum_{r=1}^{j/2-3}\frac{2^{2r-1}}{(2^{r+j/2}+1)(2^{2r-1}+2^{r+j/2}+1)}\int_{e_0}P\\&\geqslant 2^{-j-7}\left(\frac{j}{2}-3\right)\int_{e_0}P\geqslant Mj2^{-j},\end{aligned}$$

where M is an absolute positive constant. Finally using (45), we obtain

$$|\gamma^{-1}(1/N_j)\,s_{N_j}(f_j,0)|\geqslant M\,\frac{\log N_j/N_j}{|\gamma(1/N_j)|}=M_j\xrightarrow[j\to\infty]{}\infty.$$

Next we consider the following substitute for (27): a function $f\in L(\mathbb{T})$ is said to belong to $\mathscr{L}_{r,\alpha}(x_0)$, $\alpha>0$, $x_0\in\mathbb{R}$, if there exists a polynomial P of order r, $0\leqslant r\leqslant\alpha$, such that

$$\frac{1}{h}\int_{|x-x_0|\leqslant h}|f(x)-P(x-x_0)|\,dx=O(h^\alpha).$$

The class $\mathscr{L}_{r,\alpha}(x_0)$ was introduced by Calderon and Zygmund [1].

THEOREM 3.2. *Let φ satisfy (4), $\psi\in C^{(m)}(\mathbb{R})$ with $\psi^{(\ell)}$ bounded for $\ell\leqslant m$, $|\psi(x)|\leqslant\mathscr{C}/(1+|x|^n)$, $n>1$. If $f\in\mathscr{L}_{r,\alpha}(x_0)$, $\alpha>0$, $r\leqslant m$, $r<n-1$, then*

$$|s_N(f,x_0)-P(0)|=O(N^{-\min\{\alpha,n-1\}}),\qquad N\to\infty,\tag{46}$$

for $\alpha\neq n-1$, and

$$|s_N(f,x_0)-P(0)|=O(N^{-\alpha}\log N),\qquad N\to\infty,\tag{47}$$

for $\alpha=n-1$.

Proof. Let $\alpha\neq n-1$. It is clear that x_0 is a Lebesgue point of f. Then $s_N(f,x_0)$ converges to $P(0)$ (see, e.g., [15]). On the basis of this and (16), it suffices to prove that

$$s_{2^j+L}(f,x_0)-s_{2^j-1}(f,x_0)=O(2^{-j\min\{\alpha,n-1\}})\tag{48}$$

for all $j=0, 1, ..., L=0, 1 \cdots 2^j-1$. Using (18) and Lemmas 2.2 and 2.3, we have

$$|s_{2^j+L}(f, x_0)-s_{2^j-1}(f, x_0)|$$

$$\leqslant 2^j \int_{-\infty}^{\infty} |f(x)-P(x-x_0)| \sum_{\nu \in \mathbb{Z}} |\psi(2^j x+\nu)| \, |\psi(2^j x_0+\nu)| \, dx$$

$$\leqslant \mathscr{C}(n)\, 2^j \int_{-\infty}^{\infty} |f(x)-P(x-x_0)| \, \mu(2^j(x-x_0)) \, dx, \tag{49}$$

where $\mu(u)=\mathscr{C}/(1+|u|^n)$. Let j_0 be a positive integer such that

$$\frac{1}{h} \int_{|x-x_0| \leqslant h} |f(x)-P(x-x_0)| \, dx \leqslant \mathscr{C} h^{\alpha}, \qquad 0<h \leqslant 2^{j_0}.$$

The monotonicity of μ implies that for all $j \geqslant j_0$

$$|s_{2^j+L}(f, x_0)-s_{2^j-1}(f, x_0)|$$

$$\leqslant 2^{j+1} \mu(0) \int_{|x-x_0| \leqslant 2^{-j}} |f(x)-P(x-x_0)| \, dt$$

$$+\sum_{k=-j+1}^{-j_0} 2^{j+1} \mu(2^{j+k}) \int_{|x-x_0| \leqslant 2^k} |f(x)-P(x-x_0)| \, dt$$

$$+\sum_{k=-j_0+1}^{\infty} 2^{j+1} \mu(2^{j+k}) \int_{|x-x_0| \leqslant 2^k} |f(x)-P(x-x_0)| \, dt$$

$$=\Sigma_0+\Sigma_1+\Sigma_2. \tag{50}$$

The sums Σ_0, Σ_1 can be estimated as well as the similar ones in (34). Thus, we obtain

$$\Sigma_0+\Sigma_1=O(2^{-j \min\{\alpha, n-1\}}). \tag{51}$$

Using the evident relations

$$\int_{|x-x_0| \leqslant h} |f(x)-P(x-x_0)| \, dt=O(h), \qquad 0<h \leqslant 1,$$

$$\int_{|x-x_0| \leqslant h} |f(x)-P(x-x_0)| \, dt=O(h^{r+1}), \qquad h \geqslant 1,$$

we have

$$\Sigma_2 = O(1)\left(\sum_{k=-j_0+1}^{-1} 2^{j+k}\mu(2^{j+k}) + \sum_{k=0}^{\infty} 2^{j+k(r+1)}\mu(2^{j+k})\right)$$
$$+ O(1)\left(2^{j(1-n)} \sum_{k=-j_o+1}^{\infty} 2^{k(1-n)} + 2^{j(1-n)} \sum_{k=0}^{\infty} 2^{k(r-n+1)}\right)$$
$$= O(2^{-j\min\{\alpha, n-1\}}).$$

This together with relations (49)–(51) implies (48). The case $\alpha = n-1$ is similar. ▮

Remark. In the hypotheses of Theorem 3.2 the smoothness of ψ can be replaced by (18).

4. EXPANSIONS WITH RESPECT TO BI-ORTHOGONAL WAVELET SYSTEMS

In this section we extend Theorem 3.1 to a wide class of wavelets (not necessary generated by a non-periodic scaling function). We shall use the definition of PMRA and general approach to periodic wavelets given on the basis of this definition in [15]. To be more precise, we will briefly review the basic details of this approach.

Let us introduce more notations. If f is a periodic function, then $S_j f = f(\cdot + 2^{-j})$. If $f \in L(\mathbb{T})$, then $\omega_n^j f$ is a 1-periodic function defined by its Fourier series

$$\sum_{\ell \in \mathbb{Z}} \hat{f}(2^j \ell + n)\, e^{2\pi i (2^j \ell + n)x}.$$

DEFINITION 4.1. Let $X = L_p(\mathbb{T})$, $1 \leqslant p < \infty$ or $X = C(\mathbb{T})$, $V_j \subset X$, $j = 0, 1, \ldots$. The collection $\{V_j\}_{j=0}^{\infty}$ is called a PMRA in X, if the following properties hold:

MR1. $V_j \subset V_{j+1}$, $j = 0, 1, 2, \ldots$;

MR2. $\bigcup_{j=0}^{\infty} V_j$ is dense in X;

MR3a. $\dim V_j = 2^j$, $j = 0, 1, 2, \ldots$;

MR3b. $\dim \{f \in V_j : S_j f = \lambda f\} \leqslant 1$ for all $\lambda \in \mathbb{R}$, $j = 0, 1, 2, \ldots$;

MR4a. if $f \in V_j$, then $f(2\,\cdot) \in V_{j+1}$, $j = 0, 1, 2, \ldots$;

MR4b. if $f \in V_{j+1}$, then $f(\cdot/2) + f((\cdot + 1)/2) \in V_j$, $j = 0, 1, 2, \ldots$;

MR4c. if $f \in V_j$, then $S_j f \in V_j$, $j = 0, 1, 2, \ldots$.

Let $\{V_j\}_{j=0}^{\infty}$ be a PMRA. A sequence of functions $\{G_j\}_{j=0}^{\infty}$ is said to be a *scaling sequence* (for this PMRA), if $G_j \in V_j$ and the functions $S_j^k G_j$, $k=0, ..., 2^j-1$, constitute a basis of the space V_j for all $j=0, 1, ...$.

THEOREM 4.2. *Let $X=L_p(\mathbb{T})$, $1 \leqslant p < \infty$ or $X=C(\mathbb{T})$. A sequence of functions $\{G_j\}_{j=0}^{\infty} \subset X$ is a scaling sequence for some PMRA if and only if the following properties hold*:

G1. $\hat{G}_0(k)=0$ *for all* $k \neq 0$;

G2. *for all* $j=0, 1, ...$ *and* $k=0, 1, ..., 2^j-1$ *there exists* $m \in \mathbb{Z}$ *such that* $\hat{G}_j(2^j m+k) \neq 0$;

G3. *for all* $k \in \mathbb{Z}$ *there exists* $j \in \{0, 1, ...\}$ *such that* $\hat{G}_j(k) \neq 0$;

G4. *for each* $j=0, 1, ...$ *and* $k \in \mathbb{Z}$ *there exists* $\lambda_k^j \neq 0$ *such that* $\hat{G}_j(2^j m+k)=\lambda_k^j \hat{G}_{j+1}(2^{j+1}m+2k)$ *for all* $m \in \mathbb{Z}$;

G5. *for each* $j=0, 1, ...$ *and* $k \in \mathbb{Z}$ *there exists* $\delta_k^j \in \mathbb{R}$ *such that* $\hat{G}_j(2^{j+1}m+k)=\delta_k^j \hat{G}_{j+1}(2^{j+1}m+k)$ *for all* $m \in \mathbb{Z}$.

In particular, this theorem implies that a scaling sequence exists in each PMRA. It is clear that for each scaling function φ the collection of sets V_j defined by (6) is a PMRA in the sense of Definition 4.1 and the functions Φ_{j0} defined by (5) constitute a scaling sequence for this PMRA.

Let us consider pairs of PMRA with the first component $\{V_j\}_{j=0}^{\infty}$ in L_p, $1 \leqslant p < \infty$ and the second one $\{\tilde{V}_j\}_{j=0}^{\infty}$ in L_q, $1/p+1/q=1$ (C for $p=1$) and call them (p, q)*-pairs of PMRA*.

PROPOSITION 4.3. *If $(\{V_j\}_{j=0}^{\infty}, \{\tilde{V}_j\}_{j=0}^{\infty})$ is a (p, q)-pair of PMRA and $\{G_j\}_{j=0}^{\infty}$ and $\{\tilde{G}_j\}_{j=0}^{\infty}$ are scaling sequences for $\{V_j\}_{j=0}^{\infty}$ and $\{\tilde{V}_j\}_{j=0}^{\infty}$, respectively, then the shift bases $\{S_j^n G_j\}_{n=0}^{2^j-1}$ in V_j and $\{S_j^n \tilde{G}_j\}_{n=0}^{2^j-1}$ in $\tilde{V}_j$ are bi-orthonormal if and only if*

$$\langle \omega_n^j G_j, \omega_n^j \tilde{G}_j \rangle = 2^{-j}. \tag{52}$$

Let $(\{G_j\}_{j=0}^{\infty}, \{\tilde{G}_j\}_{j=0}^{\infty})$ be scaling sequences of a (p, q)-pair of PMRA $(\{V_j\}_{j=0}^{\infty}, \{\tilde{V}_j\}_{j=0}^{\infty})$ and let H_j, $\tilde{H}_j$, $j=0, 1, ...$ be 1-periodic functions with the Fourier coefficients $\hat{H}_j(r)$, $\hat{\tilde{H}}_j(r)$, $r \in \mathbb{Z}$, respectively, defined by

$$\hat{H}_j(r)=e^{\pi i 2^{-j} r}\overline{\tilde{\delta}_{2^j+r}^j}\,\hat{\tilde{G}}_{j+1}(r), \qquad \hat{\tilde{H}}_j(r)=e^{\pi i 2^{-j} r}\overline{\delta_{2^j+r}^j}\,\hat{G}_{j+1}(r),$$

where δ_k^j, $\tilde{\delta}_k^j$ are the factors from Theorem 4.2 for the sequences $\{G_j\}_{j=0}^{\infty}$, $\{\tilde{G}_j\}_{j=0}^{\infty}$, respectively. The functions H_j, $\tilde{H}_j$ are called *wavelet functions* and the spaces

$$W_j=\operatorname{span}\{S_j^n H_j, n=0, ..., 2^j-1\},$$

$$\tilde{W}_j=\operatorname{span}\{S_j^n \tilde{H}_j, n=0, ..., 2^j-1\}$$

are called *wavelet spaces* of the (p, q)-pair $(\{V_j\}_{j=0}^{\infty}, \{\tilde{V}_j\}_{j=0}^{\infty})$.

THEOREM 4.4. *Let* $(\{V_j\}_{j=0}^{\infty}, \{\tilde{V}_j\}_{j=0}^{\infty})$ *be a* (p, q)*-pair of PMRA, and let* $\{G_j\}_{j=0}^{\infty}$, $\{\tilde{G}_j\}_{j=0}^{\infty}$ *be their scaling sequences,* $\{H_j\}_{j=0}^{\infty}$, $\{\tilde{H}_j\}_{j=0}^{\infty}$ *are corresponding wavelet function sequences,* W_j, $j=0, 1, \ldots$ *are wavelet spaces. If* (52) *holds, then*

H1. $W_j \subset V_{j+1}$;

H2. *each function* $f \in V_j$, $j=1, 2, \ldots$ *can be represented in the form* $f=f_1+f_2$, *where* $f_1 \in V_{j-1}$, $f_2 \in W_{j-1}$;

H3. $\langle S_{j_1}^{n_1} H_{j_1}, S_{j_2}^{n_2} \tilde{H}_{j_2} \rangle = 0$ *for all* $j_1, j_2 = 0, 1, \ldots, j_1 \neq j_2$, $n_1 = 0, \ldots, 2^{j_1}-1$, $n_2 = 0, \ldots 2^{j_2}-1$;

H4. $\langle S_j^{n_1} H_j, S_j^{n_2} \tilde{H}_j \rangle = \delta_{n_1, n_2}$ *for all* $j=0, 1, \ldots, n_1, n_2 = 0, \ldots, 2^j-1$.

Next we fix a (p, q)-pair $(\{V_j\}_{j=0}^{\infty}, \{\tilde{V}_j\}_{j=0}^{\infty})$ satisfying the hypothesis of Theorem 4.4. For each $f \in L_p$ we can consider the following wavelet expansion:

$$\langle f, \tilde{G}_0 \rangle G_0 + \sum_{j=0}^{\infty} \sum_{n=0}^{2^j-1} \langle f, S_j^n \tilde{H}_j \rangle, S_j^n H_j. \tag{53}$$

This double series can be transformed to a single one as was done for (7). Let s_N denote the partial sum of this single series. Notice that, due to Theorem 4.4,

$$s_{2^j-1}(f) = \sum_{n=0}^{2^j-1} \langle f, S_j^n \tilde{G}_j \rangle S_j^n G_j. \tag{54}$$

THEOREM 4.5. *Let* $f \in \mathscr{L}_\alpha(x_0)$, $\alpha > 0$. *If for all* $x \in [-1/2, 1/2]$

$$|G_j(x)| \leqslant \mathscr{C} \frac{2^{j(1-\kappa)}}{1+(2^j |x|)^m}, \qquad |\tilde{G}_j(x)| \leqslant \mathscr{C} \frac{2^{j\kappa}}{1+(2^j |x|)^m}, \qquad m>1, \tag{55}$$

then

$$s_{2^j-1}(f, x_0) - s = O(2^{-j \min\{\alpha, m-1\}}), \qquad j \to \infty, \tag{56}$$

if $\alpha \neq m-1$,

$$s_{2^j-1}(f, x_0) - s = O(j 2^{-j\alpha}), \qquad j \to \infty, \tag{57}$$

if $\alpha = m-1$. *If, moreover, for all* $x \in [-1/2, 1/2]$

$$|H_j(x)| \leqslant \mathscr{C} \frac{2^{j(1-\lambda)}}{1+(2^j |x|)^n}, \qquad |\tilde{H}_j(x)| \leqslant \mathscr{C} \frac{2^{j\lambda}}{1+(2^j |x|)^n}, \qquad n>1, \tag{58}$$

then

$$s_N(f, x_0) - s = O(N^{-\min\{\alpha, n-1\}}), \qquad N \to \infty, \tag{59}$$

if $\alpha \neq n-1$;

$$s_N(f, x_0) - s = O(N^{-\alpha} \log N), \qquad N \to \infty, \tag{60}$$

if $\alpha = n-1$.

Proof. Consider the case $\alpha \neq n-1$, $\alpha \neq m-1$. First we note that, by (55), (58), the functions $\tilde{G}_j$, $\tilde{H}_j$ are bounded. Hence the sums $s_N(f)$ can be considered for each $f \in L(\mathbb{T})$. By Theorems 4.2 and 4.4, $s_N(h) = h$ for all $h \equiv \text{const}$ and all $N = 0, 1\dots$. Then

$$s_{2^j-1}(f, x_0) - s = \int_0^1 (f(t) - s) \sum_{k=0}^{2^j-1} S_j^k G_j(x_0) \overline{S_j^k \tilde{G}_j(t)}\, dt. \tag{61}$$

The functions $S_j^k G_j$, $S_j^k \tilde{G}_j$ can be represented in the form

$$S_j^k G_j = \sum_{\ell \in \mathbb{Z}} g_j(2^j x + 2^j \ell + k), \qquad S_j^k \tilde{G}_j = \sum_{\ell \in \mathbb{Z}} \tilde{g}_j(2^j x + 2^j \ell + k),$$

where

$$g_j(t) = \begin{cases} G_j(2^{-j} t), & \text{if} \quad t \in [-2^{j-1}, 2^{j-1}], \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{g}_j(t) = \begin{cases} \tilde{G}_j(2^{-j} t), & \text{if} \quad t \in [-2^{j-1}, 2^{j-1}], \\ 0, & \text{otherwise}. \end{cases}$$

Then, by (55),

$$|g_j(t)| \leqslant \mathscr{C} \frac{2^{j(1-\kappa)}}{1 + |t|^m}, \qquad |\tilde{g}_j(t)| \leqslant \mathscr{C} \frac{2^{j\kappa}}{1 + |t|^m}.$$

Hence applying Lemmas 2.2 and 2.3 to (61), we have

$$|s_{2^j-1}(f, x_0) - s| \leqslant 2^j \int_{-\infty}^{\infty} |f(t) - s|\, \mu(2^j(t - x_0))\, dt,$$

where $\mu(u) = \mathscr{C}/(1 + |u|^m)$. The right-hand side in this inequality is the same as it was in (32). Hence to prove (56), it remains to repeat the arguments of the proof of Theorem 3.1 based on (32). Now we assume that (58) holds. By (56), the sums $s_{2^j-1}(f, x_0)$ converge to s as $j \to \infty$. On the basis of this and (16), for (56) it suffices to prove that

$$s_{2^j-1}(f, x_0) - s_{2^j+L}(f, x_0) = O(2^{-j \min\{\alpha, n-1\}})$$

for all $j=0, 1, ..., L=0, 1, ..., 2^j-1$. By Theorem 4.4,

$$\int_0^1 S_j^k \tilde{H}_j = \int_0^1 G_0 S_j^k \tilde{H}_j = 0$$

for all $j=0, 1, ..., k=0, ..., 2^j-1$. This implies that

$$s_{2^j-1}(f, x_0) - s_{2^j+L}(f, x_0) = \int_0^1 (f(t)-s) \sum_{k=0}^{L} S_j^k H_j(x_0) \overline{S_j^k \tilde{H}_j(t)}\, dt.$$

Similarly to the proof of (56) we can conclude from this that

$$|s_{2^j-1}(f, x_0) - s_{2^j+L}(f, x_0)| \leqslant 2^j \int_{-\infty}^{\infty} |f(t)-s|\, \mu(2^j(t-x_0))\, dt,$$

where $\mu(u) = \mathscr{C}/(1+|u|^n)$. Again it remains to repeat the arguments used for Theorem 3.1. The cases $\alpha = n-1$, $\alpha = m-1$ are similar. ∎

The assumption $m>1$ in (55) cannot be relaxed. Moreover, if we consider this assumption separately for the first and the second PMRA with m_1 and m_2, respectively, then neither $m_1>1$ nor $m_2>1$ can be relaxed. Let $(\{V_j\}_{j=0}^{\infty}, \{\tilde{V}_j\}_{j=0}^{\infty})$ be the pair of PMRA defined as follows. The second component $\{\tilde{V}_j\}_{j=0}^{\infty}$ is a PMRA generated by the scaling function φ defined by (42). The functions

$$\tilde{G}_j(x) = \sum_{\ell \in \mathbb{Z}} \varphi(2^j x + 2^j \ell) = 2^{-j} \sum_{k \in \mathbb{Z}} \hat{\varphi}(2^{-j}k)\, e^{2\pi i k x}, \qquad j=0, 1, ...$$

constitute a scaling sequence in $\{\tilde{V}_j\}_{j=0}^{\infty}$. We define the space V_j as the linear span of the functions $S_j^n G_j$, where

$$G_j(x) = \sum_{k=-2^{j-1}}^{2^{j-1}}{}' \overline{(\hat{\varphi}(2^{-j}k))^{-1}}\, e^{2\pi i k x}.$$ [2]

It is clear that $\{V_j\}_{j=0}^{\infty}$ is a PMRA and $\{G_j\}_{j=0}^{\infty}$ is a scaling sequence of this PMRA. Orthonormal wavelet system of this PMRA was investigated in [2]. Since for each $j=0, 1, ...$ the sequences $S_j^k G_j$, $S_j^k \tilde{G}_j$ satisfy (52), they are bi-orthonormal and the corresponding wavelet systems $S_j^k H_j$, $S_j^k \tilde{H}_j$ are bi-orthonormal too, due to Theorem 4.4. The functions G_j, $\tilde{G}_j$ are, evidently,

[2] The symbol $\sum'$ means a sum with the factors 1/2 in the first and the last terms.

in $L_\infty(\mathbb{T})$. Hence for any $f \in L(\mathbb{T})$ we can consider both the wavelet expansions, (53) and

$$\langle f, G_0 \rangle \tilde{G}_0 + \sum_{i=0}^{\infty} \sum_{n=0}^{2^j-1} \langle f, S_j^n H_j \rangle S_j^n \tilde{H}_j. \tag{62}$$

Let $\tilde{s}_N(f)$ denote the Nth partial sum of this series. Since φ is compactly supported, (55) holds for $\tilde{G}_j$ with any $m > 1$. It is not difficult to check that

$$|G_j(y)| \leqslant \frac{1 + \mathscr{C}\, 2^j}{2^j\, |y|}.$$

Thus, (55) holds for G_j with $m = 1$, $\kappa = 0$. It is possible to prove that for an arbitrary $\gamma(u) = o(1)$, $u \to \infty$ there exist functions $f_1, f_2 \in L(\mathbb{T})$ supported on $[1/4, 1/2]$ such that

$$\gamma^{-1}(2^j)\, s_{2^j-1}(f_1, 0) \xrightarrow[j \to \infty]{} \infty,$$
$$\gamma^{-1}(2^j)\, \tilde{s}_{2^j-1}(f_2, 0) \xrightarrow[j \to \infty]{} \infty.$$

On the other hand, the series in (53), (62) converge to zero for each $f \in \mathscr{L}_\alpha(0)$, $\alpha > 0$.

5. DISCRETE WAVELET FOURIER TRANSFORM

In this section we consider wavelets generated by a compactly supported scaling function. Such wavelets are very important for various applications, in particular for the reconstruction of functions. There exists a fast scheme for the computation of the wavelet Fourier coefficients of a reconstructed function (subband filtering scheme). The algorithm is based on the following arguments. Due to (2), (3), the coefficients of the jth level (corresponding to the basis elements of the spaces V_j, W_j) can be expressed by the coefficients of the $(j-1)$th level. This allows recursive formulas to be obtained. The smaller the support of a scaling function, the faster is the process of the computation by these formulas. However, recursive processes accumulate errors from level to level. Thus, for large j computations can give wrong results. We propose an alternative algorithm for the reconstruction of functions based on wavelet expansions. The idea is to replace wavelet Fourier coefficients by their discrete analogs, which can be computed without any recursion. In other words, we are going to introduce a wavelet analog of DFT (the discrete Fourier transform for the trigonometric system). It is well known that for a smooth function its DFT is close to the corresponding Fourier coefficient, but the former is more preferable for numerical problems.

Further we fix a PMRA $\{V_j\}_{j=0}^{\infty}$ generated by a scaling function φ such that supp $\varphi \in [-R, R]$. Set

$$K_j(x, y) = \sum_{\ell=0}^{2^j-1} \Phi_{j\ell}(x)\, \overline{\Phi_{j\ell}(y)},$$

where $\Phi_{j\ell}$ are functions defined by (5).

THEOREM 5.1. *For each $j = 0, 1, ...$ there exist $y_{ks} \in [0, 1]$, $k = 0, ..., 2^j - 1$, $s = 1, ..., M$, where M depends only on R, and constants $\alpha_1, ..., \alpha_M$ such that for all $f \in V_j$ and all $x \in [0, 1]$*

$$f(x) = 2^{-j} \sum_{k=0}^{2^j-1} \sum_{s=1}^{M} \alpha_s f(y_{ks})\, K_j(x, y_{ks}). \tag{63}$$

Our proof of this theorem is based on the following auxiliary statements.

LEMMA 5.2. *For all $f_1, ..., f_N \in L[0, 1]$, there exist $y_1, ..., y_M \in [0, 1]$ and $\alpha_1, ..., \alpha_M \in \mathbb{R}$, $M \leqslant N$, such that*

$$\sum_{i=1}^{M} \alpha_i f_j(y_i) = \int_0^1 f_j, \qquad j = 1 \cdots N. \tag{64}$$

Proof. If there exist $y_1, ..., y_N \in [0, 1]$ such that $\det\{f_j(y_i)\}_{i,j=1}^{N} \neq 0$, then $\alpha_1, ..., \alpha_N$ can be found as a solution of the linear system (64). Otherwise, the dimension of the space of the vectors $r(y) = (f_1(y), ..., f_N(y))$, $y \in [0, 1]$ is equal to $M < N$. Thus, there exist $y_1, ..., y_M \in [0, 1]$ and $k_1, ..., k_M \in \{1, ..., N\}$ such that $\det\{f_{k_j}(y_i)\}_{i,j=1}^{M} \neq 0$, $f_{k_l}(y_i) = \sum_{j=1}^{M} \times \beta_{lj} f_{k_j}(y_i)$, $i = 1, ..., M$, $l = M+1, ..., N$. Let $\alpha_1, ..., \alpha_M$ be a solution of the system

$$\sum_{i=1}^{M} \alpha_i f_{k_j}(y_i) = \int_0^1 f_{k_j}, \qquad j = 1 \cdots M.$$

Since the vectors $r(y_i)$, $i = 1, ..., M$, constitute a basis, for each $y \in [0, 1]$ there exist numbers $\lambda_j(y)$ such that $f_j(y) = \sum_{i=1}^{M} \lambda_i(y) f_j(y_i)$, $j = 1, ..., N$. Hence, for all $\ell > M$ we have

$$\int_0^1 f_{k_l}(y)\, dy = \int_0^1 \sum_{i=1}^{M} \lambda_i(y) f_{k_l}(y_i)\, dy = \int_0^1 \sum_{i=1}^{M} \lambda_i(y) \sum_{j=1}^{M} \beta_{lj} f_{k_j}(y_i)\, dy$$

$$= \sum_{j=1}^{M} \beta_{lj} \int_0^1 f_{k_j}(y)\, dy = \sum_{i=1}^{M} \alpha_i \sum_{j=1}^{M} \beta_{lj} f_{k_j}(y_i) = \sum_{i=1}^{M} \alpha_i f_{k_l}(y_i). \quad \blacksquare$$

LEMMA 5.3. *Let j be a positive integer, $x, t \in [0, 1]$, $y_k = 2^{-j}(t+k)$, $k = 0, ..., 2^j - 1$. Then*

$$2^{-j} \sum_{k=0}^{2^j-1} |K_j(x, y_k)| \leqslant \mathscr{C}.$$

Proof. Since φ is compactly supported, for all j large enough

$$\begin{aligned}
2^{-j} &\sum_{k=0}^{2^j-1} |K_j(x, y_k)| \\
&\leqslant \sum_{k=0}^{2^j-1} \sum_{n=0}^{2^j-1} \sum_{\ell_1 \in \mathbb{Z}} |\varphi(2^j x + 2^j \ell_1 + n)| \sum_{\ell \in \mathbb{Z}} |\varphi(2^j y_k + 2^j \ell + n)| \\
&\leqslant \sum_{k=0}^{2^j-1} \sum_{n=0}^{2^j-1} \sum_{\ell_1 \in \mathbb{Z}} |\varphi(2^j x + 2^j \ell_1 + n)| \sum_{\ell \in \mathbb{Z}} |\varphi(t + k + 2^j \ell_1 + n + 2^j \ell)| \\
&\leqslant \sum_{\ell \in \mathbb{Z}} \sum_{k=0}^{2^j-1} \sum_{n \in \mathbb{Z}} |\varphi(2^j x + n)\, \varphi(t + k + n + 2^j \ell)| \\
&\leqslant \sum_{k=0}^{2^j-1} \sum_{n \in \mathbb{Z}} |\varphi(2^j x + n)\, \varphi(t + k + n)|.
\end{aligned}$$

It remains to note that, by Lemma 2.3,

$$\sum_{k=0}^{2^j-1} \sum_{n \in \mathbb{Z}} |\varphi(2^j x + n)\, \varphi(t + k + n)| \leqslant \mathscr{C} \sum_{k \in \mathbb{Z}} \mu\left(\frac{2^j x - t - k}{4}\right) \leqslant \mathscr{C} \sum_{k \in \mathbb{Z}} \mu(k),$$

where μ is an even compactly supported majorant of φ decreasing on $[0, \infty]$. ∎

Proof of Theorem 5.1. Consider the functions $h_{\nu\mu}(t) = \varphi(t+\nu)\, \overline{\varphi(t+\mu)}$, $\nu, \mu \in \mathbb{Z}$ defined on $[0, 1]$. Since $h_{\nu\mu} \not\equiv 0$ only if $-1 - R \leqslant \nu$, $\mu \leqslant R$, by Lemma 5.2, there exist $t_s \in [0, 1]$, $\alpha_s \in \mathbb{R}$, $s = 1, ..., M$, such that

$$\sum_{s=1}^{M} \alpha_s h_{\nu\mu}(t_s) = \int_0^1 h_{\nu\mu} \tag{65}$$

for all $\mu, \nu \in \mathbb{Z}$. Let S denote the right-hand side of (63) with $y_{ks} = 2^{-j}(k + t_s)$. Since both f and S are elements of the space V_j, it suffices to prove that

$$\int_0^1 f\overline{\Phi_{jn}} = \int_0^1 S\overline{\Phi_{jn}} \tag{66}$$

for all $j=0, 1, ..., n=0, ..., 2^j-1$. If $a_n=\int_0^1 f\overline{\Phi_{jn}}$, then $f=\sum_{n=0}^{2^j-1} a_n\Phi_{jn}$ and

$$\int_0^1 S\overline{\Phi_{jn}}=2^{-j}\sum_{m=0}^{2^j-1} a_m \sum_{k=0}^{2^j-1}\sum_{s=1}^{M}\alpha_s \sum_{r=0}^{2j-1}\Phi_{jm}(y_{ks})\,\overline{\Phi_{jr}(y_{ks})}\int_0^1 \overline{\Phi_{jn}}\Phi_{jr}$$

$$=2^{-j}\sum_{m=0}^{2^j-1} a_m \sum_{k=0}^{2^j-1}\sum_{s=1}^{M}\alpha_s\Phi_{jm}(y_{ks})\,\overline{\Phi_{jn}(y_{ks})}=\sum_{m=0}^{2^j-1} a_m A_{nm}.$$

Next, by (65),

$$A_{nm}=\sum_{s=1}^{M}\alpha_s\sum_{k=0}^{2^j-1}\sum_{l_1\in\mathbb{Z}}\varphi(t_s+2^jl_1+k+m)\sum_{l\in\mathbb{Z}}\overline{\varphi(t_s+2^jl+k+n)}$$

$$=\sum_{s=1}^{M}\alpha_s\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\varphi(t_s+k)\,\overline{\varphi(t_s+2^jl+k+n-m)}$$

$$=\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\int_0^1\varphi(t+k)\,\overline{\varphi(t+k+2^jl+(n-m))}\,dt$$

$$=\sum_{l\in\mathbb{Z}}\int_{-\infty}^{\infty}\varphi(t)\,\overline{\varphi(t+2^jl+(n-m))}\,dt.$$

Since $\{\varphi(t+\nu)\}_{\nu\in\mathbb{Z}}$ is an orthonormal system on $\mathbb{R}$, A_{nm} does not vanish only for $n=m$, and $A_{nn}=1$. Hence $\sum_{m\in\mathbb{Z}} a_m A_{nm}=a_n$. This yields Theorem 5.1. ∎

COROLLARY 5.4. *If $f, g\in V_j$, then*

$$\langle f, g\rangle=2^{-j}\sum_{k=0}^{2^j-1}\sum_{s=1}^{M}\alpha_s f(y_{ks})\,g(y_{ks}). \tag{67}$$

Proof. Equality (67) follows immediately from (63) if we take into account that

$$\int_0^1 g(t)\,K_j(t, y)\,dt=g(y). \quad ∎$$

We now consider an arbitrary 1-periodic function f defined at each point. Set

$$\sigma_j(f, x)=2^{-j}\sum_{k=0}^{2^j-1}\sum_{s=1}^{M}\alpha_s f(y_{ks})\,K_j(x, y_{ks}).$$

THEOREM 5.5. *If $f\in C$, then*

$$\|f-\sigma_j(f)\|_\infty\leqslant \mathscr{C}E_{2^j}(f)_\infty, \tag{68}$$

Proof. Let g be an element of best approximation in V_j to f. Since, due to (63), $\sigma_j(g, x) = g$, we easily obtain (68) from Lemma 51. ∎

This theorem, together with Theorem 2.1, shows that $\sigma_j(f)$ is a good tool of approximation for smooth functions. (67) implies that $\sigma_j(f)$ can be represented as the wavelet expansion

$$\sigma_j(f) = \sum_{n=0}^{2^j-1} c_n(f)\, w_n,$$

where

$$c_n(f) = 2^{-j} \sum_{k=0}^{2^j-1} \sum_{s=1}^{M} \alpha_s f(y_{ks})\, w_n(y_{ks}), \qquad n = 0, ..., 2^j - 1.$$

A coefficient $c_n(f)$ will be called a discrete wavelet Fourier transform (DWFT) of f.

THEOREM 5.6. *If* $f \in C$, $j = 0, 1, ...,$ $\ell = 0, ..., j-1,$ $r = 0, ..., 2^\ell - 1,$ $n = 2^\ell + r$, *then*

$$|c_n(f) - \langle f, w_n \rangle| \leqslant \mathscr{C} 2^{-\ell/2} E_{2^j}(f)_\infty. \tag{69}$$

Proof. Let g be an element of best approximation in V_j to f. By (67), $c_n(g) = \langle g, w_n \rangle$. Hence

$$\begin{aligned} |c_n(f) - \langle f, w_n \rangle| &\leqslant |c_n(f-g) - (f-g, w_n\rangle| \\ &\leqslant E_{2^j}(f)_\infty \left(2^{-j} \sum_{k=0}^{2^j-1} \sum_{s=1}^{M} |\alpha_s|\, |w_n(y_{ks})| + \int_0^1 |w_n| \right). \end{aligned}$$

Since $w_n(y_{ks}) = \int_0^1 w_n(t)\, K_j(t, y_{ks})\, dt$, due to Lemma 5.3,

$$|c_n(f) - \langle f, w_n \rangle| \leqslant \mathscr{C} E_{2^j}(f)_\infty \int_0^1 |w_n|. \tag{70}$$

By the definition of w_n,

$$\int_0^1 |w_n(t)|\, dt \leqslant 2^{\ell/2} \int_0^1 \sum_{m \in \mathbb{Z}} |\psi(2^\ell t + 2^\ell m + r)|\, dt = 2^{-\ell/2} \int_{-\infty}^{\infty} |\psi(t)|\, dt.$$

Combining this with (70), we obtain (69). ∎

We see that the wavelet Fourier coefficients of a smooth function can be replaced by DWFTs in problems of reconstruction of functions and compression of information. The algorithm of computation is very simple, DWFT can be computed just by the definition. Finding the numbers α_s and the nodes y_{ks} is not difficult in practice. We can take arbitrary points $t_1, ..., t_M \in [0, 1]$ with $M = 2([R]+1)^2$ and check if the determinant of the system (66) does not vanish. If these points are not suitable, we can try to succeed with other ones. Let us estimate complexity of this algorithm. Suppose the values α_s, $w_n(y_{ks})$ are known and count the number of operations that we need to compute all the coefficients $c_n(f)$, $n = 0, ..., 2^j - 1$. Set $n = 2^\ell + r$, $\ell < j$, $r = 0, ..., 2^\ell - 1$, $2^\ell > 4R$. Since φ is supported on $[-R, R]$, the wavelet function ψ is supported on $[-2R, 2R]$ (see [3, Chap. 5]). Hence $w_n(x) \neq 0$ only in the following cases:

(a) $r < 2R$, $0 \leqslant x \leqslant 2^{-\ell}(2R - r)$, $1 - 2^{-\ell}(2R + r) \leqslant x \leqslant 1$;

(b) $2R \leqslant r \leqslant 2^\ell - 2R$, $1 - 2^{-\ell}(2R + r) \leqslant x \leqslant 1 - 2^{-\ell}(r - 2R)$;

(c) $r \geqslant 2^\ell - 2R$, $0 \leqslant x \leqslant 1 - 2^{-\ell}(r - 2R)$, $2 - 2^{-\ell}(2R + r) \leqslant x \leqslant 1$.

So, only $4R2^{-\ell}$ terms of the sum $\sum_{k=0}^{2j-1} w_n(y_{ks}) f(y_{ks})$ do not vanish. Therefore, to compute $c_n(f)$ we should make at most $8RM2^{j-\ell}$ operations. This implies that the complexity of the whole algorithm does not exceed $8RM(1+j)\,2^j$.

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